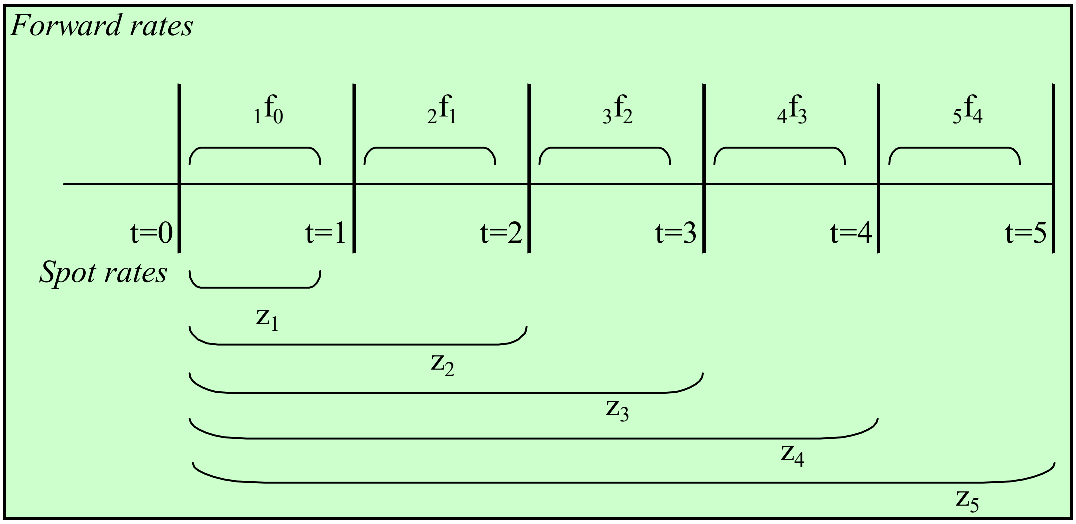
1. Spot and Forward Interest Rates. Bootstrapping

1.1 Money market

The spot rate is defined as the discounting rate for a cash flow at a specific maturity. It is the rate of interest for the period from now to a specific moment in the future. Interest for the cash flow is calculated in arrears. WIBOR and LIBOR are example of spot rates. A yield for a Treasury bill is a spot rate, but discount rate for a Treasury bill is not a spot rate (interest is calculated in advance).

A forward rate is used to calculate interest between two moments in the future. Interest for the cash flow is also calculated in arrears.

Market forward rates exist for such instruments as FRA or eurodollar futures.



It is also possible to calculate implied (theoretical, "fair" forward rates). For money market instruments the following formula shows the relation between forward rate and two spot rates:

(1) $$\left(1+{}_tf_{t-1}\frac{(d_t-d_{t-1})}{360}\right)\left(1+z_{t-1}\frac{d_{t-1}}{360}\right)=\left(1+z_t\frac{d_t}{360}\right)$$

where:
$_tf_{t-1}$ - forward rate,
z_t - spot rate,
d_t - number of days from today to maturity in moment t.

Implied forward rates may be derived using the following formulas:

(2) $${}_tf_{t-1}=\left[\frac{\left(1+\frac{z_td_t}{360}\right)}{\left(1+\frac{z_{t-1}d_{t-1}}{360}\right)}-1\right]\frac{360}{d_t-d_{t-1}}\text{, or }\quad {}_tf_{t-1}=\frac{\left[\frac{z_td_t-z_{t-1}d_{t-1}}{d_t-d_{t-1}}\right]}{\left(1+\frac{z_{t-1}d_{t-1}}{360}\right)}$$

Problem 1. WIBOR implied forward rates

Day 03-10-03. WIBOR and WIBID quotations are following:

Maturity	Days	WIBOR	WIBID	midpoint
O/N	1	5,45%	5,18%	5,32%
T/N	2	5,46%	5,19%	5,33%
1W	7	5,40%	5,20%	5,30%
2W	14	5,38%	5,20%	5,29%
1M	31	5,34%	5,20%	5,27%
3M	92	5,29%	5,15%	5,22%
6M	183	5,24%	5,05%	5,15%
9M	274	5,20%	5,00%	5,10%
1Y	366	5,15%	4,96%	5,06%

Source: onet.pl

FRA quotations:

FRA	Bid	Ask
1 X 4	5,15%	5,20%
2 X 5	5,13%	5,18%
3 X 6	5,03%	5,08%
6 X 9	4,85%	4,90%
9 X 12	4,85%	4,90%

Source: Treasury Management Services

Required

(a) Calculate 3x6, 6x9, 9x12 implied forward rates.

(b) Compare implied forward rates with market forward rates.

Solution

(a)

We are using the following formula (convention a/365).

$$ {}_{t}f_{t-1} = \left[\frac{\left(1 + \frac{z_t d_t}{365}\right)}{\left(1 + \frac{z_{t-1} d_{t-1}}{365}\right)} - 1 \right] \frac{365}{d_t - d_{t-1}} $$

	z_t bid z_{t-1} ask	z_t ask z_{t-1} bid	z_t midpoint z_{t-1} midpoint
3 X 6	4,74%	5,26%	5,00%
6 X 9	4,40%	5,37%	4,88%
9 X 12	4,09%	5,39%	4,74%

(b)

Market spread is very narrow. Spread between implied forward rates is wide.

1.2 Capital market

Spot rate as a function of forward rates

Spot rate for a period T may be defined as a product of forward rates for all periods[1]:

(3) $(1+z_T)^T = (1+{}_1f_0)(1+{}_2f_1)\dots(1+{}_Tf_{T\text{-}1})$

gdzie:
${}_tf_{t\text{-}1}$ – forward rate,
z_T – spot rate for the period T,
T – number of periods.

Spot rate is equal to geometric average of forward rates calculated as:

(4) $z_T = \sqrt[T]{(1+{}_1f_0)(1+{}_2f_1)\dots(1+{}_Tf_{T\text{-}1})} - 1$

With continuous compounding the arithmetic average of forward rates is used:

(5) $e^{Tz_T^*} = e^{{}_1f_0^*}e^{{}_2f_1^*}\dots e^{{}_Tf_{T\text{-}1}^*} = e^{{}_1f_0^*+{}_2f_1^*+\dots+{}_Tf_{T\text{-}1}^*} = e^{\sum_{t=0}^{T-1}{}_{t+1}f_t^*}$

and

(6) $z_T^* = \dfrac{{}_1f_0^*+{}_2f_1^*+\dots+{}_Tf_{T\text{-}1}^*}{T} = \dfrac{\sum_{t=0}^{T-1}{}_{t+1}f_t^*}{T}$

Discounting factor

The price of a zero coupon bond with a face value of 1 (discounting factor) with discrete compounding is:

(7) $P = \dfrac{1}{(1+z_T)^T} = \dfrac{1}{(1+{}_1f_0)(1+{}_2f_1)\dots(1+{}_Tf_{T\text{-}1})}$

With continuous compounding it is equal to:

(8) $P = \exp[-Tz_T^*] = \exp[-({}_1f_0^*+{}_2f_1^*+\dots+{}_Tf_{T\text{-}1}^*)] = \exp\left[-\sum_{t=0}^{T-1}{}_{t+1}f_t^*\right]$

As the forward price for a zero coupon bond for any future period is:

(9) ${}_{t+1}F_t = \dfrac{1}{1+{}_{t+1}f_t}$ (discrete compounding)

(10) ${}_{t+1}F_t = \exp[-{}_{t+1}f_t^*]$ (continuous compounding)

the price of a zero coupon bond with a face value of 1 is equal to the product of forward prices:

(11) $P = {}_1F_0\ {}_2F_1 \dots {}_TF_{T\text{-}1}$

[1] For the first period forward rate is assumed to be the firsts spot rate ${}_1f_0=z_1$. This notation was introduced by Hicks. See. J.R.Hicks, *Value and Capital*, 2 ed., Oxford University Press, London, 1942, p. 141-145.

Problem 2. Spot and forward rates for a zero coupon bond

The spot rates for a zero coupon bonds are following:

Period	Spot rate
1	5%
2	6%
3	7%
4	8%

Required:

(a) Calculate compounding factors, discounting factors and forward rates.

(b) Calculate continuously compounded spot rates and appropriate compounding factors.

(c) Calculate continuously compounded forward rates, and the appropriate spot rates.

(d) Compute spot prices of zero coupon bonds using:

(i). continuously compounded spot rates,

(ii). continuously compounded forward rates,

(e) Calculate forward prices of zero coupon bonds.

Calculate prices of zero coupon bonds using forward prices.

Solution

(a)

t	z_t	$(1+z_T)^T$	$\frac{1}{(1+z_T)^T}$	${}_t f_{t-1}$
1	5%	105,00%	95,24%	5,00%
2	6%	112,36%	89,00%	7,01%
3	7%	122,50%	81,63%	9,03%
4	8%	136,05%	73,50%	11,06%

(b) (c)

t	$z_T^* = \ln(1+z_T)$	$e^{Tz_T^*}$	${}_t f_{t-1}{}^* = \ln(1+{}_t f_{t-1})$	$z_T^* = \frac{\sum_{t=0}^{T-1} {}_{t+1}f_t^*}{T}$
1	4,88%	105,00%	4,88%	4,88%
2	5,83%	112,36%	6,77%	5,83%
3	6,77%	122,50%	8,64%	6,77%
4	7,70%	136,05%	10,49%	7,70%

(d) (e)

t	$P = \exp\left[-Tz_T^*\right]$	$P = \exp\left[-\sum_{t=0}^{T-1} {}_{t+1}f_t^*\right]$	${}_{t+1}F_t = \exp\left[-{}_{t+1}f_t^*\right]$	$P = \prod_{t=0}^{T-1} {}_{t+1}F_t$
1	95,24%	95,24%	95,24%	95,24%
2	89,00%	89,00%	93,45%	89,00%
3	81,63%	81,63%	91,72%	81,63%
4	73,50%	73,50%	90,04%	73,50%

The general relation between forward rate and two spot rates is

(12) $(1+z_y)^y = (1+z_x)^x (1+{}_y f_x)^{y-x}$

so implied forward rate may be calculated using the following formula

(13) $${}_y f_x = \left[\frac{(1+z_y)^y}{(1+z_x)^x}\right]^{\frac{1}{y-x}} - 1$$

where ${}_y f_x$ is the implied annualized forward rate from maturity x to maturity y in years.

Using the three-year spot rate and two-year spot rate, one-year rate two years forward) can be developed:

(14) $${}_3 f_2 = \frac{(1+z_3)^3}{(1+z_2)^2} - 1$$

Using the three-year spot rate and one-year spot rate, two-year rate one year forward) can be developed:

(15) $${}_3 f_1 = \sqrt{\frac{(1+z_3)^3}{(1+z_1)^1}} - 1$$

Problem 3. Implied forward rates for Treasury Bills

Spot interest rates for Treasury securities in Poland (3.10.2003) are following:

Maturity	Spot
1	5,086%
2	5,246%

Calculate one-year rate one year forward.

Solution

$${}_2 f_1 = \frac{(1+5{,}246\%)^2}{(1+5{,}086\%)^1} - 1 = 5{,}406\%$$

Problem 4. Implied LIBOR forward rate

Eurodollar futures contract with 30 days maturity is equal to 98,845. LIBOR for 30 day deposits is equal to 1,120%, and for 120 days deposits is 1,160%.

Questions

(a) What is the implied LIBOR forward rate ?

(b) Comparing this rate with futures rate, should investor buy or sell futures contract ?

(c) What is investor's rate of return for 120 days if he borrows money for 30 days ? and sells futures contract?

Solution

(a)

The relation between spot rates and forward rate is:

$$\left(1+\frac{r_2 t_2}{360}\right)=\left(1+\frac{r_1 t_1}{360}\right)\left(1+\frac{{}_2f_1(t_2-t_1)}{360}\right)$$

Implied forward rate is :

$${}_2f_1=\left[\frac{\left(1+\frac{r_2 t_2}{360}\right)}{\left(1+\frac{r_1 t_1}{360}\right)}-1\right]\frac{360}{t_2-t_1} \qquad = \qquad 1{,}172\%$$

(b)

He should sell, because futures rate 1,155% is lower than 1,172%.

(c) Return is:

$$i_2=\left\{\left(1+\frac{r_1 t_1}{360}\right)\left(1+\frac{{}_2f_1(t_2-t_1)}{360}\right)-1\right\}\frac{360}{t_2} \qquad = \qquad 1{,}147\%$$

((1+1,120% * 30 / 360) (1+1,155% * 90 / 360) - 1)(360/ 120)

1.3 Bootstrapping

The process of creating a theoretical spot curve from coupon securities.

Problem 5. Bootstrapping

Coupon rates and yields to maturity for government securities (with semiannual interests) are following:

Maturity (years)	Coupon rate	YTM
0,5	0,0%	6,0%
1,0	0,0%	6,4%
1,5	6,0%	7,0%
2,0	7,0%	7,5%
2,5	9,0%	7,8%
3,0	8,0%	8,1%

Required

(a) Calculate market prices, spot rates and forward rates.

(b) Compare the prices for a three-year bond

I. cash flows discounted with spot rates (non arbitrage approach)

II. cash flows discounted with YTM (traditional approach)

(c) Calculate interest with forward rates and use spot rates to establish the price of the three-year bond with floating interest rate.

Solution

(a)

t	c_t	YTM	P_M	z_t	$1/(1+z_t/2)^t$	${}_t f_{t-1}$
1	0,0000%	3,0000%	97,09	6,0000%	0,971	6,0000%
2	0,0000%	3,2000%	93,89	6,4000%	0,939	6,8008%
3	3,0000%	3,5000%	98,60	7,0226%	0,902	8,2733%
4	3,5000%	3,7500%	99,09	7,5471%	0,862	9,1287%
5	4,5000%	3,9000%	102,68	7,8766%	0,824	9,1997%
6	4,0000%	4,0500%	99,74	8,1891%	0,786	9,7588%

Theoretical spot rates are derived from observed YTM.

$$97{,}09 = \frac{100}{(1+\frac{z_1}{2})^1}, \text{ or } z_1 = 6{,}0000\%$$

$$93{,}89 = \frac{0}{(1+0{,}03)^1} + \frac{100}{(1+\frac{z_2}{2})^2}, \text{ or } z_2 = 6{,}4000\%$$

$$98{,}60 = \frac{3}{(1+0{,}03)^1} + \frac{3}{(1+0{,}032)^2} + \frac{103}{(1+\frac{z_3}{2})^3}, \text{ or } z_3 = 7{,}0226\%$$

$$99{,}09 = \frac{3{,}5}{(1+0{,}03)^1} + \frac{3{,}5}{(1+0{,}032)^2} + \frac{3{,}5}{(1+0{,}035113)^3} + \frac{103{,}5}{(1+\frac{z_4}{2})^4},$$

or $z_4 = 7{,}5471\%$, etc.

Implied forward rates are calculated as follows:

$${}_1f_0 = 6{,}0000\%$$

$${}_2f_1 = 2\left[\frac{(1+3{,}2000\%)^2}{(1+3{,}0000\%)^1} - 1\right] = 6{,}8008\%$$

$${}_3f_2 = 2\left[\frac{(1+3{,}5113\%)^3}{(1+3{,}2000\%)^2} - 1\right] = 8{,}2733\%$$

$${}_4f_3 = 2\left[\frac{(1+3{,}7735\%)^4}{(1+3{,}5113\%)^3} - 1\right] = 9{,}1287\%$$

itd.

(b)
Both methods give the same price.

		Disc. rate = Spot rates		Disc. rate = YTM	
t	Cash flows	Disc. factor	Disc. flows	Disc. factor	Disc. flows
1	4,0000%	0,971	3,8835%	0,961	3,8443%
2	4,0000%	0,939	3,7558%	0,924	3,6947%
3	4,0000%	0,902	3,6066%	0,888	3,5509%
4	4,0000%	0,862	3,4492%	0,853	3,4126%
5	4,0000%	0,824	3,2975%	0,820	3,2798%
6	104,0000%	0,786	81,7458%	0,788	81,9560%
		Σ	99,7383%	Σ	99,7383%

(c)
The sum of interest calculated with forward rates and the face value discounted using spot rates is equal to the present face value (or 100%).

t	Cash flows	Disc. factor	Disc. flows
1	3,0000%	0,971	2,9126%
2	3,4004%	0,939	3,1928%
3	4,1367%	0,902	3,7298%
4	4,5643%	0,862	3,9358%
5	4,5998%	0,824	3,7920%
6	104,8794%	0,786	82,4370%
		Σ	100,0000%

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