# 1. Interest Rate Exposure. Stochastic Methods

## 1.1 Exposure

Book value
Book value + commitments
Market Value
Economic value (valuation models)

### Valuation Models

1. Discrete models
    a. Binomial
    b. One-path (DCF models)
        i. Traditional – one discount rate
        ii. Non arbitrage – many discount rates (spot rates)
2. Continuous models (BSM) – what can be valued ?

Option pricing (binomial and BS) models are used to value assets with embedded options. Binomial model is used to value callable bonds, putable bonds, floating rate notes, and structured notes in which the cash flows are based on interest rate. Simulation methods are used to value assets in assumed risky environment. The Monte Carlo simulation is used to value mortgage-backed securities and certain type of asset-backed securities in which the cash flows are based on interest rate path.

1. Traditional Valuation:

$$PV = \frac{CF_1}{(1+RRR)^1} + \frac{CF_2}{(1+RRR)^2} + \ldots + \frac{CF_n}{(1+RRR)^n} + \frac{CV_n}{(1+RRR)^n}$$

2. Non Arbitrage Valuation

$$PV = \frac{CF_1}{(1+RRR_1)^1} + \frac{CF_2}{(1+RRR_2)^2} + \ldots + \frac{CF_n}{(1+RRR_n)^n} + \frac{CV_n}{(1+RRR_n)^n}$$

Differences between two approaches are shown in the following table (bond as an example):

| | Fixed rate | Floating rate |
|---|---|---|
| Traditional approach | $P = \frac{cB}{(1+YTM)^1} + \frac{cB}{(1+YTM)^2} + \ldots + \frac{cB+B}{(1+YTM)^T}$ | $P = \frac{z_1 B}{(1+YTM)^1} + \frac{{}_2f_1 B}{(1+YTM)^2} + \ldots + \frac{{}_T f_{T-1} B + B}{(1+YTM)^T}$ |
| Arbitrage - free approach | $P = \frac{cB}{(1+z_1)^1} + \frac{cB}{(1+z_2)^2} + \ldots + \frac{cB+B}{(1+z_T)^T}$ | $P = \frac{z_1 B}{(1+z_1)^1} + \frac{{}_2f_1 B}{(1+z_2)^2} + \ldots + \frac{{}_T f_{T-1} B + B}{(1+z_T)^T}$ |

P - price, c - coupon rate, B - face value, YTM - yield to maturity, z - spot rate, f - forward rate.

The traditional valuation methodology discounts every cash flow of an asset by the same discount rate. The arbitrage-free approach values an asset with each cash flow discounted at its unique discount rate (spot rate).

**Problem 1. Bond price, YTM and maturity**

Consider a coupon bond with a face value $1000 paying an annual coupon of 10%.
Required:
(a) Calculate the market price of a bond as a result of changes in market yield in the range 8-12% and for different maturities: 1 year, 5 years, 10 years, 15 years 20 years, 25 years and 30 years.
(b) Show sensitivity of return changes on above yields and maturities.

**Solution**

Ad 1.
Market prices of a bond as a result of changes in market yield for the assumed maturities are:

| | Maturity (years) | | | | | | |
|---|---|---|---|---|---|---|---|
| Rate | 1 | 5 | 10 | 15 | 20 | 25 | 30 |
| 8% | 1018,5 | 1079,9 | 1134,2 | 1171,2 | 1196,4 | 1213,5 | 1225,2 |
| 9% | 1009,2 | 1038,9 | 1064,2 | 1080,6 | 1091,3 | 1098,2 | 1102,7 |
| 10% | 1000,0 | 1000,0 | 1000,0 | 1000,0 | 1000,0 | 1000,0 | 1000,0 |
| 11% | 991,0 | 963,0 | 941,1 | 928,1 | 920,4 | 915,8 | 913,1 |
| 12% | 982,1 | 927,9 | 887,0 | 863,8 | 850,6 | 843,1 | 838,9 |

Ad 2.
The percentage price changes are following:

| | Maturity (years) | | | | | | |
|---|---|---|---|---|---|---|---|
| Rate | 1 | 5 | 10 | 15 | 20 | 25 | 30 |
| 8% | 1,9% | 8,0% | 13,4% | 17,1% | 19,6% | 21,3% | 22,5% |
| 9% | 0,9% | 3,9% | 6,4% | 8,1% | 9,1% | 9,8% | 10,3% |
| 10% | 0,0% | 0,0% | 0,0% | 0,0% | 0,0% | 0,0% | 0,0% |
| 11% | -0,9% | -3,7% | -5,9% | -7,2% | -8,0% | -8,4% | -8,7% |
| 12% | -1,8% | -7,2% | -11,3% | -13,6% | -14,9% | -15,7% | -16,1% |

## 1.2 Stochastic models

### 1.2.1 General equilibrium Term Structure Models

Cox, Ingersoll and Ross [1985] derived stochastic process of interest rates as[1]:

(1) $$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dz$$

where
r – the current spot rate,
θ - the central location or long term value,
κ - the pull parameter that governs the speed at which the spot rate is drawn back to the long term value,
σ - volatility,
dt – a small change in time,
dz - standard, one-dimensonal Wiener process.

[1] J.C.Cox, J.E. Ingersoll, Jr., Stephen Ross, *An Intertemporal General Equilibrium Model of Asset Prices*, „Econometrica”, 1985, vol. 53, no. 2 s. 363-384 oraz tychże *Theory of the Term Structure of Interest Rates*, „Econometrica”, 1985, vol. 53, no. 2 s. 385-407.

### 1.2.2 Arbitrage-free Modelling

These models take a linear stochastic differential equation of the general form

(2) $dr = \mu(r,t)dt + \sigma(r,t)dz$

gdzie:
r - the current spot rate,
μ - the drift term,
σ - volatility,
dt - a small change in time,
dz - standard, one-dimensonal Wiener process.

**Ho** and **Lee** provided one of the first arbitrage-free models of the term structure. The stochastic differential equation was[2]:

(3) $dr = \mu(t)dt + \sigma dz$

They assumed that interest rate shocks were normally distributed. The mean μ was selected to match exactly the current structure. The volatility parameter σ was fixed. The disadvantage of this model is that negative interest rates are possible. In binomial lattice, the up and down jumps were expressed as

(4) $r_u = r_0 + \mu(ts) + \sigma\sqrt{ts}$

(5) $r_d = r_0 + \mu(ts) - \sigma\sqrt{ts}$

gdzie:
ts – time step.

**In the lognormal model** (the same assumptions as in the BSM)) stochastic differential equation was:

(6) $dr = \mu(t)\, r\, dt + \sigma\, r\, dz$

or (using Ito's lemma):

(7) $d\ln(r) = \left[\mu(t) - \frac{\sigma^2}{2}\right]dt + \sigma dz$

This model excludes possibility of obtaining negative, but still ignores the strong mean-reverting process. The interest rate volatility is proportional to rate level but is still independent in time. The up and down jumps were expressed as follows

(8) $r_u = r_0 \exp^{\mu(ts)+\sigma\sqrt{ts}}$

(9) $r_d = r_0 \exp^{\mu(ts)-\sigma\sqrt{ts}}$

**Black, Derman and Toy** adopted a lognormal distribution and introduced time-varying rate volatility. The stochastic differential equation was[3]:

(10) $dr = \mu(t)\, r\, dt + \sigma(t)\, r\, dz$

[2] T.S.Y. Ho, S. Lee, *Term Structure Movements and Pricing Interest Rate Contingent Claims*, „Journal of Finance", 1986, vol. 41, no. 5.

[3] F. Black, E. Derman, W. Toy, *A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options*, „Financial Analysts Journal", January 1990, vol. 46, no. 1, s. 33-39.

The up and down jumps were expressed as follows:

(11) $r_u = r_0 \exp^{\mu(ts)+\sigma(ts)\sqrt{ts}}$

(12) $r_d = r_0 \exp^{\mu(ts)-\sigma(ts)\sqrt{ts}}$

**Blacka** and **Karasiński** extended the previous model by explicitly incorporating a mean reversion parameter κ[4]

(13) $dr = \kappa(t)\,\{\ln[\mu(t)] - \ln[r(t)]\}\, r\, dt + \sigma(t)\, r\, dz$

where:

κ - mean reversion parameter.

The up and down jumps were expressed as follows:

(14) $r_u = r_0 \exp^{\kappa(ts)[\mu(ts) - r(ts)]ts+\sigma(ts)\sqrt{ts}}$

(15) $r_d = r_0 \exp^{\kappa(ts)[\mu(ts) - r(ts)]ts-\sigma(ts)\sqrt{ts}}$

**Hull** and **White'a**[5] introduced general framework:

(16) $dx = a\left[\frac{\theta(t)}{a} - x\right]dt + \sigma\, dz$

**Heatha, Jarrow** and **Morton**[6] assumes that each forward rate may change based on its own sensitivities to the underlying factors. The term structure may change and twist in a wide variety of ways. The stochastic differential equation of the family of forward rates can be expressed as

(17) $df(T) = \int_0^t \mu(v,T,\omega)dv + \sum_{i=1}^{n} \int_{i=1}^{t} \sigma_i(v,T,\omega)dW_i(v)$

To compare stochastic differential models the following general formula may be used:

(18) $dr(t) = [\alpha_1(t) + \alpha_2(t)r(t) + \alpha_3(t)\ln(r(t))]d(t) + [\beta_1(t) + \beta_2(t)r(t)]^{\gamma}\, dz$

Tabela 1. Interest Rate Term Structure Models

| Autor | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\beta_1$ | $\beta_2$ | $\gamma$ |
|---|---|---|---|---|---|---|
| Merton (1974) | ☺ | | | ☺ | | 1 |
| Vasicek (1977) | ☺ | ☺ | | ☺ | | 1 |
| Brennan-Schwartz (1979) | ☺ | ☺ | | | ☺ | 1 |
| Cox-Ingersoll-Ross (1980) | | | | | ☺ | 1,5 |
| Cox-Ingersoll-Ross (1985) | ☺ | ☺ | | | ☺ | 0,5 |
| Ho-Lee (1986) | ☺ | | | ☺ | | 1 |
| Salomon Brothers | | ☺ | | | ☺ | 1 |
| Black-Derman-Toy | | ☺ | | | ☺ | 1 |
| Black-Karasiński (1991) | | ☺ | ☺ | | ☺ | 1 |
| Pearson-Sun (1994) | ☺ | ☺ | | ☺ | ☺ | 0,5 |

Source: Table based on idea presented in the book: A.Weron, R.Weron, *Inżynieria finansowa. Wycena instrumentów pochodnych. Symulacje komputerowe. Statystyka rynku*, Wydawnictwa Naukowo-Techniczne, Warszawa 1998, s. 211.

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4 Por. F. Black i P. Karasiński, *Bond and Option Pricing When Short Rates Are Lognormal*, „Financial Analysts Journal", 1991, vol. 47, no. 4.

5 J. Hull i A. White, *Using Hull-White Interest Rate Trees*, „Journal of Derivatives", 1996, vol. 3, no. 3.

6 D.Heath, R.Jarrow, A.Morton, *Bond Pricing and the Term Structure of Interest Rates: A New Methodology*, „Econometrica", 1996, vol. 60, no. 1, s. 77-105.