Options. Pricing. Binomial models. Black-Scholes model. Greeks

1. Binomial model,
2. Black-Scholes model, assumptions, modifications (dividends, currency options, options on futures)
3. Implied volatility
4. Sensitivity measures

The binomial model has the advantage of allowing to price American options. This is a discrete time model. Scenarios are shown on a binomial tree. The process of valuing an option is often called risk-neutral valuation. The BSM model is a continuous time model used to price only European options.

Simple Binomial Pricing Model

[Cox, Ross and Rubinstein]

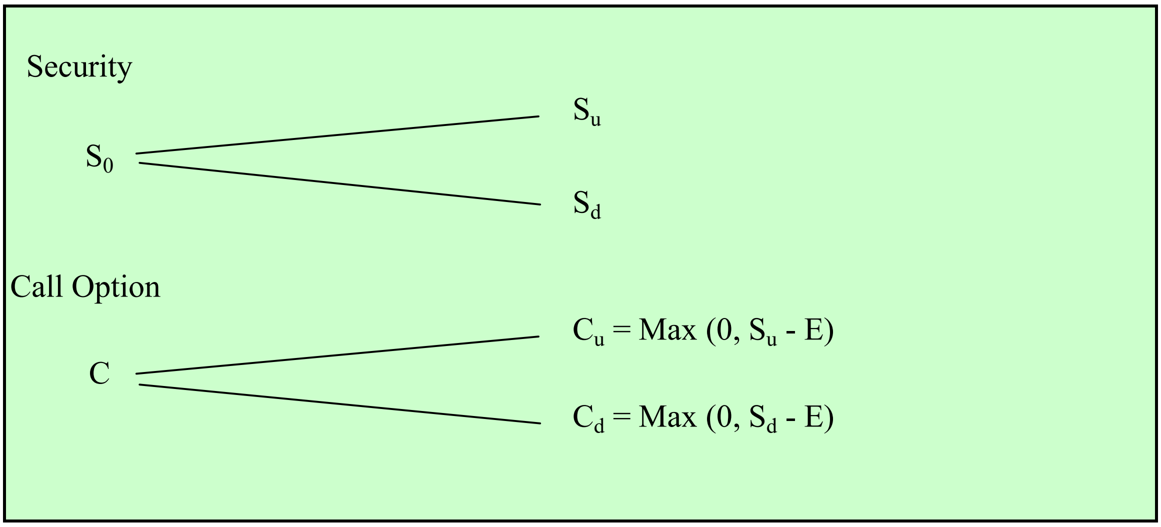


Figure. 1. Simple Binomial Pricing Model. Call Option

$$(1) \qquad C = \frac{q\,C_u + (1 - q)\,C_d}{(1 + r\,T)}$$

$$(2) \qquad q = \frac{S_0\,(1 + r\,T) - S_d}{S_u - S_d}$$

Two-Period Binomial Pricing Model

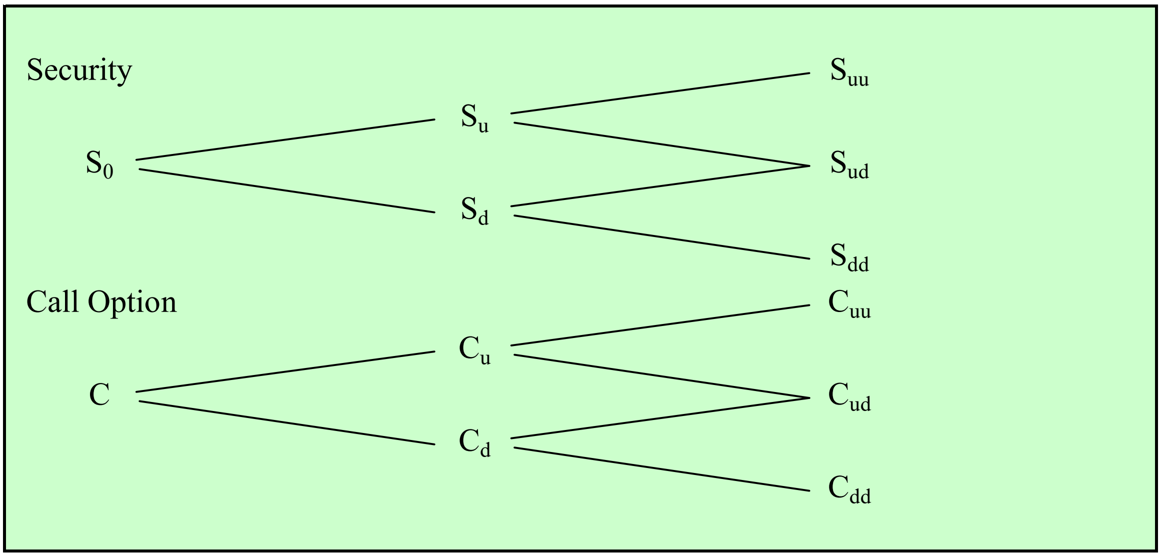


Figure 2. Two-Period Binomial Pricing Model. Call Option

$$C = \frac{qC_u + (1-q)C_d}{(1+rT)} = \frac{q\left[q_u C_{uu} + (1-q_u)C_{ud}\right] + (1-q)\left[q_d C_{ud} + (1-q_d)C_{dd}\right]}{(1+rT)^2}$$

$$q_u = \frac{S_u(1+rT) - S_{ud}}{S_{uu} - S_{ud}}$$

$$q_d = \frac{S_d(1+rT) - S_{dd}}{S_{ud} - S_{dd}}$$

Black-Scholes option pricing model

According to the Black-Scholes model, the value of a call option is given as

(1) $C = SN(d_1) - Ee^{-rT}N(d_2)$

where

$$d_1 = \frac{\ln(S/E) + (r + \sigma_s^2/2)T}{\sigma_s \sqrt{T}}$$

$$d_2 = d_1 - \sigma_s \sqrt{T}$$

σ_s^2 is the variance of the asset's returns

N(x) is the cumulative probability for a unit normal variable calculated at a value of x.

Assumptions

1) Returns for the underlying asset are log normally distributed and independent over time.
2) Constant variance
3) Constant interest rate
4) No dividends
5) No early exercise

Table 1. Option Pricing. Equity and Debt as Options

	Option Pricing	Equity and Debt
C	premium	equity
S	spot price	maret value of assets
E	exercise price	book value of debt
S-C		debt
T	maturity	maturity of debt
R_B^*	risk-free rate	risk-free rate
σ_s^2	volatility of returns	volatility of ROA

Sensitivity Measures

Table 2. Sensitivity Measures

Miernik	Notacja	Call	Put
Delta	$\Delta_c = \frac{\partial C}{\partial S}$	$\Delta_c = N(d_1)$	$\Delta_p = \Delta_c - 1$
Gamma	$\gamma_c = \frac{\partial^2 C}{\partial S^2}$	$\gamma_c = \frac{n(d_1)}{S\sigma\sqrt{T}}$	$\gamma_p = \gamma_c$
Theta	$\theta_c = -\frac{\partial C}{\partial T}$	$\theta_c = \frac{-S\sigma n(d_1)}{2\sqrt{T}} - rEe^{-rT}N(d_2)$	$\theta_p = \theta_c + rEe^{-rT}$
Rho	$\rho_c = \frac{\partial C}{\partial r}$	$\rho_c = TEe^{-rT}N(d_2)$	$\rho_p = \rho_c - TEe^{-rT}$
Vega	$\upsilon_c = \frac{\partial C}{\partial \sigma}$	$\upsilon_c = S\sqrt{T}n(d_1)$	$\upsilon_p = \upsilon_c$

where: $n(d) = \frac{e^{-d^2/2}}{\sqrt{2\pi}}$

Problem 1. Binomial Pricing Model

The current security price is $100. The exercise price on the option is $110.
It will either go up to $150 or down to $90. The riskless rate of interest is 5%. Maturity is 360 days, T = 1.

(a) Calculate the price of the call option, the hedge ratio, probabilities of the up and down movements using Cox, Ross and Rubinstein model.
Compare the result with the price calculated using BSM model.
Calculate the present value of the ending payoff.

(b) Calculate the weights for the replicating strategy, the ending payoff of the call, option and the price of the call option. The bond price is $100.

Solution

(a)

Security price	Exercise price	The payoff of the call option	
150	110	C_u =	40
90	110	C_d =	0

The hedge ratio:

$$h = \frac{-(S_u - S_d)}{(C_u - C_d)} \quad = \quad -1{,}50$$

The ending payoff

$$B = S_u + h\,C_u \quad = \quad 90{,}0 \qquad\qquad B = S_d + h\,C_d \quad = \quad 90{,}0$$

$$q = \frac{S_0(1+rT) - S_d}{S_u - S_d} \quad = \quad 0{,}25 \qquad\qquad C = \frac{qC_u + (1-q)C_d}{(1+rT)} \quad = \quad 9{,}5$$

The option premium using Black-Scholes model: C_{BS} = 12,3

The present value of the ending payoff :

$$B_0 = \frac{B}{(1+rT)} \quad = \quad 85{,}7 \qquad\qquad B_0 = S_0 + h\,C \quad = \quad 85{,}7$$

(b)

$$w_a = \frac{C_u - C_d}{S_u - S_d} = -\frac{1}{h} \quad = \quad 66{,}7\%$$

$$w_b = \frac{C_d S_u - C_u S_d}{(S_u - S_d)P} \quad = \quad -60{,}0\%$$

$$C_u = w_a S_u + w_b P \quad = \quad 40$$

$$C_d = w_a S_d + w_b P \quad = \quad 0$$

$$C = w_a S_0 + w_b P_0 \quad = \quad 9{,}5$$

Problem 2. Binomial Pricing Model.

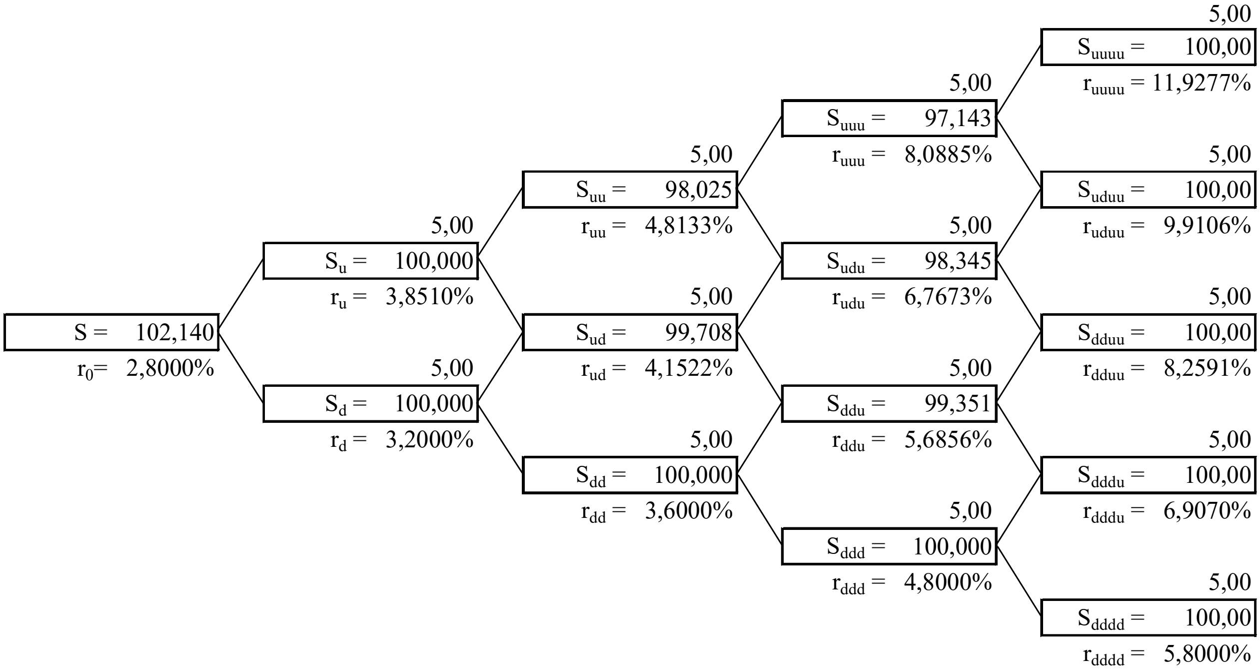
Multi-period Binomial Model. Duration and convexity

	1	2	3	4	5
T	1,00	1,00	1,00	1,00	1,00
f	2,0000%	2,4000%	2,8000%	4,0000%	5,0000%
z	2,0000%	2,1998%	2,3995%	2,7973%	3,2341%
z+shift	2,0000%	2,1998%	2,3995%	2,7973%	3,2341%
σ	10%	12%	9%	10%	10%
c	5,00%				
E		100,00	100,00	100,00	
OAS	0,80%				

Required

Calculate the price of a bond with a call option.

Calculate the effective duration and convexity.



Cena opcji 105 - 102,140 2,860

$$D = \frac{V_- - V_+}{2V_0\Delta y} \quad 2{,}447091$$

$$C = \frac{V_- + V_+ - 2V_0}{V_0(\Delta y)^2} \quad 53{,}70642$$

	102,140
-0,25%	102,799
0,25%	101,549

Problem 3. Black-Scholes Model

With the following parameteres	
S = spot price	100,00
E = exercise price	110,00
Time to expiration (number of days)	360
Risk free interest rate	5%
σ = volatility	36,1%

(a) Calculate the price of the call option using the Black-Scholes model.
(b) Calculate the price of the put option using put-call parity.

Solution

(a)

$C = S_0 N(d_1) - Ee^{-rT} N(d_2)$

$d_1 = \dfrac{\ln(S_0 / E) + (r + \sigma_s^2 / 2)T}{\sigma_s \sqrt{T}}$ $\qquad d_2 = d_1 - \sigma_s \sqrt{T}$

T	0,9863
d1	0,0511
d2	-0,3076
N(d1)	0,5204
N(d2)	0,3792
e^(-rT)	0,95
Ee^(-rT)	104,7069

C = 12,33

(b)

The price of the put option

$P = C - S_0 + Ee^{-rT} = S_0\left[N(d_1) - 1\right] - Ee^{-rT}\left[N(d_2) - 1\right]$ = 17,04

Problem 4. Implied. Volatility

S = spot price	102
E = exercise price	100
Time to expiration (number of days)	45
Risk free interest rate	7%
Call price	4
Required	
Calculate the implied volatility using sensitivity ananlysis.	

Solution

Expected volatility	Call Premium	
10%	3,28	
11%	3,38	
12%	3,49	
13%	3,61	
14%	3,72	
15%	3,84	
16%	3,97	<== above this value
17%	4,09	
18%	4,22	
19%	4,35	
20%	4,47	
21%	4,60	
22%	4,74	
23%	4,87	
24%	5,00	
25%	5,13	

Problem 5. Sensitivity Measures

The call option has the following parameters	
S = spot price	110
E = exercise price	100
Time to expiration (number of days)	30
Risk free interest rate	5%
σ = volatility	20%

(a) Calculte sensitivity measures for the call option, put option, covered call protected put, straddle, bull call spread, bear call spread.
(b) Show the sensitivity of sensitivity measures on price changes 80 - 120.

Solution

(a)

		1	2	3	4	5
		Delta	Gamma	Theta	Rho	Vega
Call Option	1	0,96	0,01	-0,02	0,08	0,03
Put Option	2	-0,04	0,01	-0,01	0,00	0,03
Covered call	3	0,04	-0,01	0,02	-0,08	0,03
Protective put	4	0,96	0,01	-0,01	0,00	0,03
Straddle	5	0,92	0,03	-0,03	0,07	0,05
Bull Call Spread	6	0,14	-0,03	0,02	0,01	-0,06
Bear Call Spread	7	-0,04	0,01	-0,01	0,00	0,02

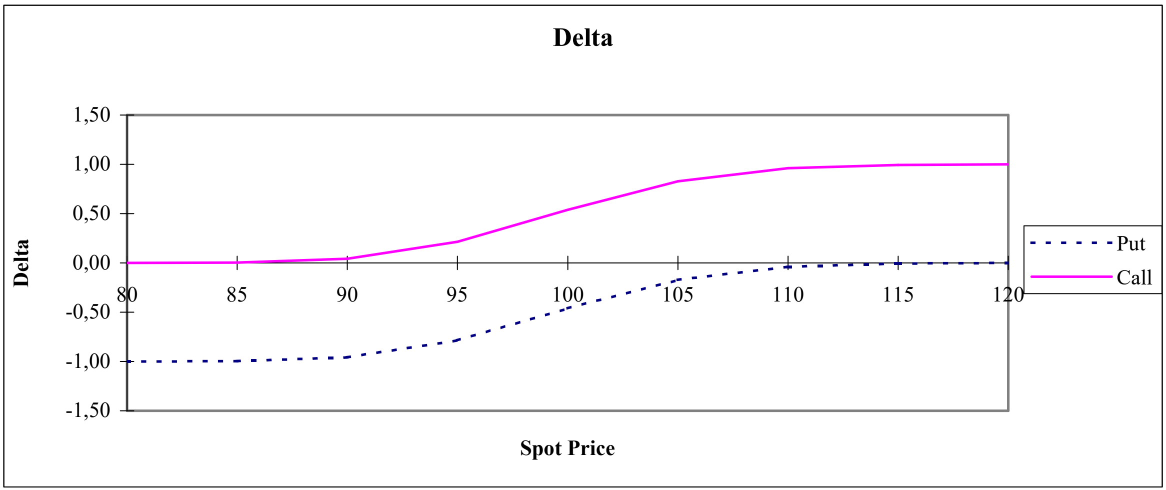
(b)

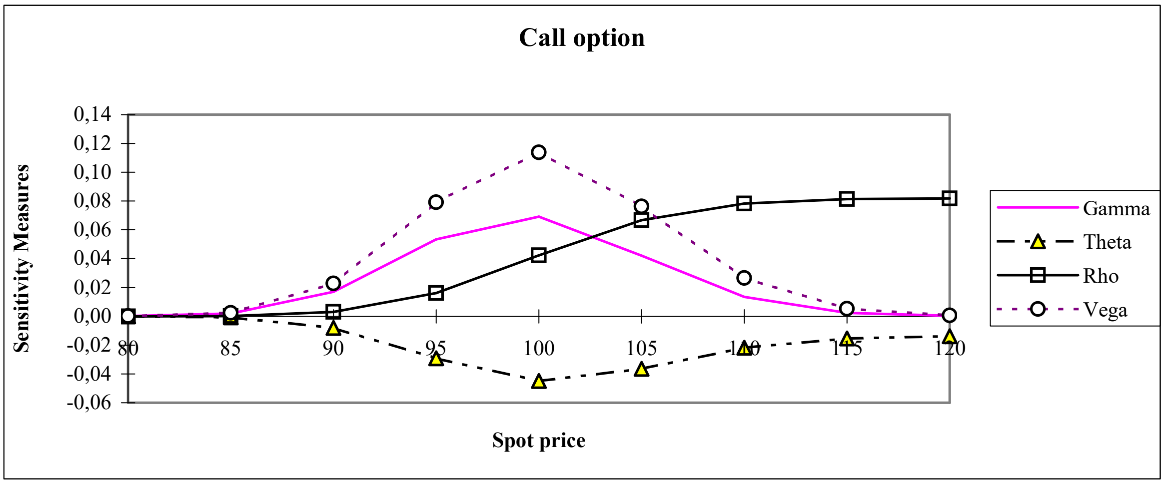
Call option

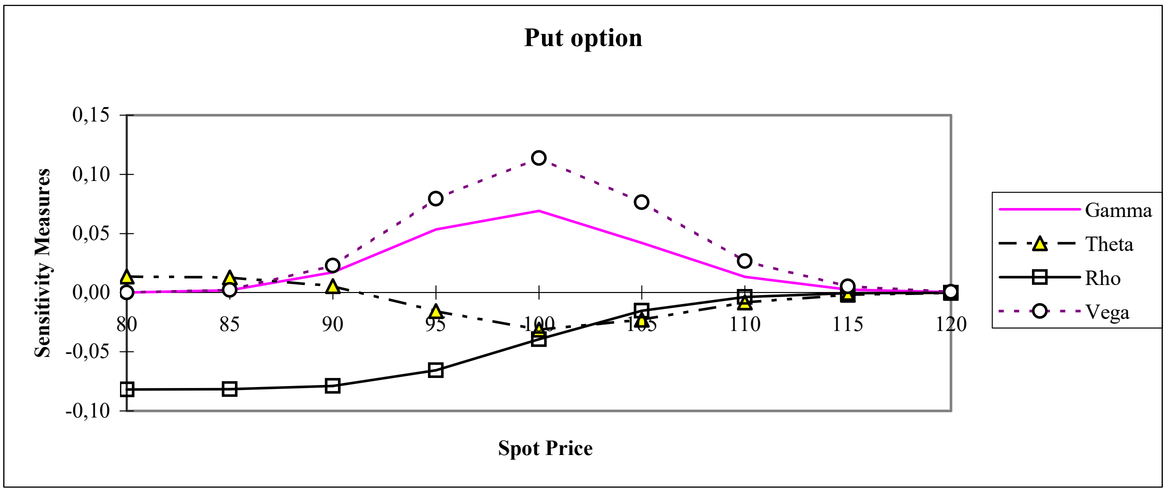
Spot Price	Delta	Gamma	Theta	Rho	Vega
80	0,00	0,00	0,00	0,00	0,00
85	0,00	0,00	0,00	0,00	0,00
90	0,04	0,02	-0,01	0,00	0,02
95	0,21	0,05	-0,03	0,02	0,08
100	0,54	0,07	-0,04	0,04	0,11
105	0,83	0,04	-0,04	0,07	0,08
110	0,96	0,01	-0,02	0,08	0,03
115	0,99	0,00	-0,02	0,08	0,01
120	1,00	0,00	-0,01	0,08	0,00

Put option

Spot Price	Delta	Gamma	Theta	Rho	Vega
80	-1,00	0,00	0,01	-0,08	0,00
85	-1,00	0,00	0,01	-0,08	0,00
90	-0,96	0,02	0,01	-0,08	0,02
95	-0,79	0,05	-0,02	-0,07	0,08
100	-0,46	0,07	-0,03	-0,04	0,11
105	-0,17	0,04	-0,02	-0,02	0,08
110	-0,04	0,01	-0,01	0,00	0,03
115	-0,01	0,00	0,00	0,00	0,01
120	0,00	0,00	0,00	0,00	0,00


Delta
1,50
1,00
0,50
0,00
-0,50
-1,00
-1,50
Delta
80
85
90
95
100
105
110
115
120
Put
Call
Spot Price


Call option
0,14
0,12
0,10
0,08
0,06
0,04
0,02
0,00
-0,02
-0,04
-0,06
Sensitivity Measures
80
85
90
95
100
105
110
115
120
Gamma
Theta
Rho
Vega
Spot price


Put option
0,15
0,10
0,05
0,00
-0,05
-0,10
Sensitivity Measures
80
85
90
95
100
105
110
115
120
Gamma
Theta
Rho
Vega
Spot Price