Pricing and Valuation of Forward and Futures

1.1 Cash-and-carry arbitrage

The price of the forward contract is related to the spot price of the underlying asset, the risk-free rate, the date of expiration, and any expected cash distributions of the underlying asset before expiration. The theoretical or "fair" price is derived from the cash-and-carry arbitrage.

Table 1. Cash-and-carry arbitrage

Strategy	Value today	Value at future time t
Strategy I Purchase the security	S	$S_t + C_t$
Strategy II Purchase a forward contract for price F	-	$S_t - F$
Invest equivalent \$ amount for settlement at time t at a risk-free interest rate r_M	S	$S\left[1+\frac{r_M t}{360}\right]$
Total value for strategy II	S	$S\left[1+\frac{r_M t}{360}\right]+S_t - F$

Source: Clarke R.G., Options and Futures: A Tutorial. The Research Foundation of The Institute of Chartered Financial Analysts 1992, s. 7.

The purchase of the security is a substitute for a risk-free investment. Both strategies begin with the same value today and result in the investor owning the asset at time t, the ending values should be equal. That is,

(1) $$S_t + C_t = S\left[1+\frac{r_M t}{360}\right]+S_t - F$$

Solving for the forward price gives

(2) $$F = S\left[1+\frac{r_M t}{360}\right] - C_t$$

For a market forward price, we may infer the implied repo rate:

(3) $$r = \left[\frac{F_M + C_t}{S} - 1\right]\frac{360}{t}$$

where:
F_M – market forward price.

Table 2. Arbitrage with low forward prices

Strategy	Value today	Value at future time t
Sell the security	-S	$-S_t - C_t$
Purchase a forward contract for price F Invest equivalent $ amount for settlement at time t at a risk-free interest rate r_M	- S	$S_t - F_M$ $S\left[1+\frac{r_M t}{360}\right]$
Total	0	$-F_M + S(1+\frac{r_M t}{360}) - C_t > 0$

An investor may sell the underlying asset and buy Treasury bills or bonds earning r_M.

The first strategy (purchase of the underlying asset) is better than the second (purchase of a forward contract) when

(4) $$F_M > F = S\left[1+\frac{r_M t}{360}\right] - C_t$$

or

(5) $$r_M < r = \left[\frac{F_M + C_t}{S} - 1\right]\frac{360}{t}$$

Table 3. Arbitrage with high forward prices

Strategy	Value today	Value at future time t
Purchase the security	+S	$+S_t + C_t$
Sell a forward contract for price F Borrow equivalent $ amount for settlement at time t at a risk-free interest rate r_M	- -S	$-S_t + F$ $-S\left[1+\frac{r_M t}{360}\right]$
Total	0	$F_M - S(1+\frac{r_M t}{360}) + C_t > 0$

The forward price with continuously compounded interest rate is:

(6) $$F = S\,e^{(r-q)\frac{t}{360}}$$

where

q – dividend yield.

1.2 Equity Index Futures Pricing

The fair price of the equity index futures contract is:

(7) $$F = S\left[1 + \frac{r_M t}{360}\right] - D$$

where
S – equity index,
r_M – annualized risk-free rate,
t – days to expiration,
D – value of dividends.

The implied repo rate is:

(8) $$r = \left[\frac{F_M + D}{S} - 1\right]\frac{360}{t}$$

1.3 Currency forward pricing

The fair price of a foreign exchange forward contract is:

(9) $$F = S_0 \frac{\left(1 + i_{Md}^{N} T\right)}{\left(1 + i_{Mf}^{N} T\right)}$$

where
S_0 – the spot exchange rate,
i_{Md}^{N} - nominal domestic interest rate,
i_{Mf}^{N} - nominal foreign interest rate.
T – time to expiration.

The implied repo rate is:

(10) $$i_d^N = \frac{\left[\frac{F_M\left(1 + i_f^N T\right)}{S_0} - 1\right]}{T}$$

1.4 Eurodollar Futures Pricing

Eurodollar futures are quoted as an index formed by subtracting from 100 the annualized three-month LIBOR (London Interbank Offered Rate) forward rate. The pricing formula for a Eurodollar futures contract is

(11) $$F = 100\,(1 - {}_{t+1}f_t)$$

where
${}_{t+1}f_t$ – is the annualized three-month LIBOR forward rate beginning at time t.

1.5 Treasury Bill Futures Pricing

The price for futures on Treasury bills is calculated as follows

(12) $F = 100\,(1 - {}_{t+1}d_t)$

where

${}_{t+1}d_t$ – the annualized three-month forward discount rate on a Treasury bill beginning at time t.

1.6 Bond Futures Pricing

The bond futures contract requires the purchase or sale of the actual Treasury bonds if it is held to maturity. The actual bond selected for delivery by the short seller is adjusted in price by a delivery factor to reflect a standardized 6 percent coupon rate. The futures price for any bond is calculated using the formula

(13) $$F = \frac{P\,(1 + \frac{r_M t}{360}) - Bc\,\frac{t + a}{365}}{f}$$

where

P – market price of bond with accrued interest,
B – par value of the bond,
r_M – interest rate,
c – annualized coupon rate,
t – days to expiration,
a – days of accrued interest,
f – delivery factor of a bond.

Problem 1. Pricing and Valuation of a forward contract

You own a stock currently worth $1000 today. You plan to sell it in 60 days.
To hedge against a possible decline in price, you enter into the forward contract to sell the security. The annualized risk-free rate is 3,5 percent.
(a) Calculate the forward price on this contract.
(b) Suppose a market dealer offers to enter into a forward contract at $1002. How you could earn an arbitrage profit.
(c) Suppose that a maket dealer offers an off-market contract. Calculate an up-front fee and indicate whether payment whether payment is made by the long to the short or vice versa.
(d After 30 days, the security sells for $1100. Calculate the gain or loss to your position.
(e) At expiration the price of the asset is $1200. Calculate the value of the forward contract.

Solution

(a)
The fair value is

$F_0 = S_0(1+i)^T - D_T$ = 1005,75

(b)

1005,75 - 1002 = 3,75

Sell the stock
Invest money
Buy forward

(c)

$V_0(0,T) = S_0 - \frac{F(0,T)}{(1+r)^{T-t}}$ = 3,73

Because the value is positive, the payment is made by the long.

(d)

$W_t = S_t - \frac{D_T}{(1+i)^{(T-t)}} - \frac{F_0^M}{(1+i)^{(T-t)}}$ = 97,13

Gain to the long position, loss to the short.

(e)

Gain on asset	1200	-	1000,00	=	200,00	
Value of forward	1200	-	1005,75	=	194,25	for long
The overall gain					5,75	

Problem 2. Pricing and Valuation of Equity Forward Contracts

The stock is selling for $100,00 and will pay a $2,00 dividend in 90 days and another $2,00 dividend in 270 days. A forward contract is expiring after 300 days. The effective annual risk-free rate is 10,00%. The actual price of equity future is 105,00.
(a) Calculate the simple interest rates for three maturities.
(b) Calculate the present value and future value od dividends using simple interest rate and effective annual rate
(c) Calculate the fair value of the equity forward using simple rates, effective annual rate and continuously compounded rate.
(d) Calculate the implied repo rate. What arbitrage positions would create this rate ?
(e) Determine the forward price and value 120 days later. The stock price is $80,00.
(f) Determine the value of the forward price at expiration date. The stock price is $102,00.

Solution

(a)

The simple rates

$$r = \frac{360}{t}\left[(1+i)^{\frac{t}{365}} - 1\right]$$

90	9,5118%	
270	9,7398%	
300	9,7785%	
360	9,8565%	
365	9,8630%	

The continuously compounded rates

$$c = \frac{365}{t}\ln\left(1+\frac{rt}{360}\right)$$

$$c = \ln(1+i)$$

90	9,5310%	These rates are different
270	9,5310%	
300	9,5310%	
360	9,5310%	
365	9,5310%	The last two rates are the same.
	9,5310%	

(b)

The present value of dividends is equal to

$$D_0 = \sum_{i=1}^{n} \frac{D_i}{\left(1+\frac{rt_i}{360}\right)} = 3,82$$

$$D_0 = \sum_{i=1}^{n} \frac{D_i}{(1+i)^{T_i}} = 3,82$$

The FV of dividends D_T = 4,13

$$\delta = \frac{365}{t}\ln\left(1+\frac{D_0}{S_0 - D_0}\right) = 4,7355\%$$

(c)
The fair forward price can be calculated as:

$$F_0 = S_0\left[1+\frac{r_M t}{360}\right] - D_0 \qquad = \qquad 104{,}02$$

or $$F_0 = [S_0 - D_0](1+i)^T \qquad = \qquad 104{,}02$$

$$F_0 = S_0(1+i)^T - D_T \qquad = \qquad 104{,}02$$

or $$F_0 = S_0 e^{(c-\delta)T} \qquad = \qquad 104{,}02$$

(d)
The implied repo rate is

$$r = \left[\frac{F_0^M + D_T}{S_0} - 1\right]\frac{360}{t} \qquad = \qquad 10{,}9542\%$$

$$i = \left(1+\frac{rt}{360}\right)^{\frac{365}{t}} - 1 \qquad = \qquad 11{,}2136\%$$

or $$i = \left(\frac{F_0^M + D_T}{S_0}\right)^{\frac{1}{T}} - 1 \qquad = \qquad 11{,}2136\%$$

or $$c = \frac{\ln\left(\frac{F_0^M + D_T}{S_0}\right)}{T} \qquad = \qquad 10{,}6283\%$$

$$c = \frac{365}{t}\ln\left(1+\frac{rt}{360}\right) \qquad = \qquad 10{,}6283\%$$

$$i = e^c - 1 \qquad = \qquad 11{,}2136\%$$

The arbitrage position would require buying a stock and selling a forward contract.
(e)
120 days later.

$$W_t = S_t - \frac{D_T}{(1+i)^{(T-t)}} - \frac{F_0^M}{(1+i)^{(T-t)}}$$

The stock price	80,00
The present value of dividends is equal to	1,92
The present value of forward price	99,24
	-21,17

or
$$W_t = \frac{S_t(1+i)^{(T-t)} - D_T}{(1+i)^{(T-t)}} - \frac{F_0^M}{(1+i)^{(T-t)}} = \frac{F_t^M - F_0^M}{(1+i)^{(T-t)}}$$

$$F_t^M \qquad = \qquad 81{,}83$$

$$W_t \qquad = \qquad -21{,}17$$

(f)

$$S_T - F_0^M \qquad = \qquad -2{,}02$$

Problem 3. Pricing of Equity Index Futures

The equity index futures price is 1742,0. The index is currently at 1662,2.
The futures contract expires in 73 days. The risk-free interest rate is 18,00%.
(a) Calculate the "fair" futures price on this contract.
(b) Calculate the implied repo rate. What arbitrage positions would create this rate.

Solution

(a)

The fair price is $$F = S\left[1 + \frac{r_M t}{360}\right] - D \quad = \quad 1722{,}87$$

(b)

The implied repo rate is $$r = \left[\frac{F_M + D}{S} - 1\right]\frac{360}{t} \quad = \quad 23{,}68\%$$

The investor should buy stocks and sell futures contract.

Problem 4. Pricing of Currency Forward

The forward exchange rate is 4,6400 zł/USD. Life of the contract: 78 days. Current spot exchange rate is: 4,5709zł/USD. Domestic risk-free interest rate is 18,00%, foreign interest rate is 6,00%.
(a) Calculate the "fair" futures price on this contract and the implied repo rate.
What arbitrage positions would create this rate.
(b) It is now 30 days later. The spot exchange rate is 4,5600.
Calculate the forward price and the value of the contract.
(c) What is the value of the forward contract at expiration. The spot exchange rate is 4,5000.

Solution

(a)

The "fair" forward exchange rate is

$$F = S_0 \frac{\left(1+i_{Md}^{N} T\right)}{\left(1+i_{Mf}^{N} T\right)} \quad = \quad 4{,}676867 \qquad 4{,}68822$$

The implied repo rate is

$$i_d^N = \frac{\left[\frac{F_M\left(1+i_f^N T\right)}{S_0} - 1\right]}{T} \qquad 13{,}07\%$$

The annualized cost rate is 13,07% plus transaction costs.
An investor should

1. borrow foreign currency at 6,00%,
2. sell currency at the current spot rate.
3. invest local currency at 18,00%,
4. buy cheap currency forward.

Ad 2.

$$W_t = \frac{S_t}{\left(1+r^f\right)^{T_1}} - \frac{F_0}{\left(1+r\right)^{T_1}}$$

4,525191	-	4,540096	=	-0,0149

Ad 3.

4,5000	-	4,6400	=	-0,1400

Problem 5. Pricing of Eurodollar Futures

Eurodollar futures contract with 30 days maturity is equal to 98,845. LIBOR for 30 day deposits is equal to 1,120%, and for 120 days deposits is 1,160%.

(a) What is the implied LIBOR forward rate ?
(b) Comparing this rate with futures rate, should investor buy or sell futures contract ?
(c) What is investor's rate of return for 120 days if he borrows money for 30 days ? and sells futures contract?

Solution

(a)

The relation between spot rates and forward rate is:

$$\left(1+\frac{r_2 t_2}{360}\right)=\left(1+\frac{r_1 t_1}{360}\right)\left(1+\frac{{}_2f_1\left(t_2-t_1\right)}{360}\right)$$

Implied forward rate is :

$${}_2f_1=\left[\frac{\left(1+\frac{r_2 t_2}{360}\right)}{\left(1+\frac{r_1 t_1}{360}\right)}-1\right]\frac{360}{t_2-t_1} \qquad = \qquad 1{,}172\%$$

(b)

He should sell, because futures rate 1,155% is lower than 1,172%.

(c) Return is:

$$i_2=\left\{\left(1+\frac{r_1 t_1}{360}\right)\left(1+\frac{{}_2f_1\left(t_2-t_1\right)}{360}\right)-1\right\}\frac{360}{t_2} \qquad = \qquad 1{,}147\%$$

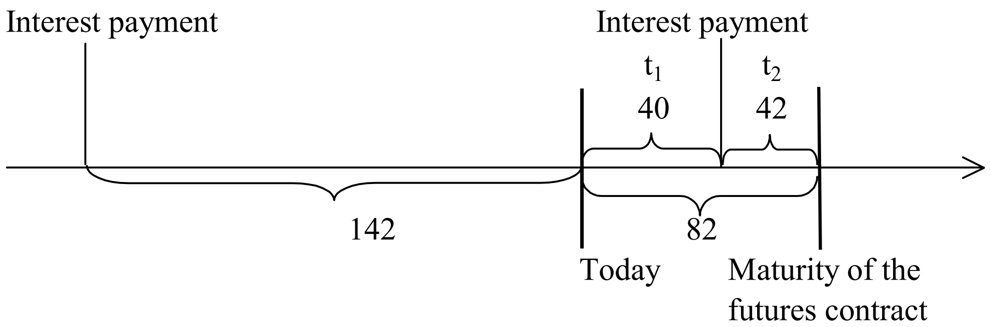
((1+1,120% * 30 / 360) (1+1,16% * 90 / 360) - 1)(360/ 120)

Problem 6. Bond Futures Pricing

The par value of a bond is 100 %, coupon rate is 10,0%, YTM = 8,71%.
Maturity is 7,5 years. Interest are paid semiannually.
The next coupon payment occurs in 40 days.
The last coupon payment was 142 days ago. The futures contract expires in 82 days. Delivery factor is equal to 1,1111.
The short-term annualized risk-free rate is 3,8%.
(a) Calculate the market price of a bond.
(b) Calculate the fair futures price and the implied repo rate when the market futures price is F =96.

Solution

(a)



The bond price with accrued interest

$$P = \frac{B\left(c\left[\left(1+\frac{y}{2}\right)^{T}-1\right]+y\right)}{y\left(1+\frac{y}{2}\right)^{T}} + \frac{Bca}{365} \quad = \quad 107 \quad + \quad 3\ 28/32 \quad = \quad 110\ 28/32$$

	3,8904	110,8803

(b)

The "fair" futures price

$$F = \frac{P\left(1+\frac{r_M t}{360}\right) - \frac{Bc}{2} - \frac{Bct_2}{365} - \frac{Bc}{2}\frac{rt_2}{360}}{f}$$

The implied repo rate

$$r = \frac{Ff + \frac{Bc}{2} + \frac{Bct_2}{365} + \frac{Bc}{2}\frac{rt_2}{360} - P}{P}\ \frac{360}{t}$$

Ff =	106,65	converted price
	5,00	interim coupon payment
	1,15	accrued interest received
	0,02	interest from reinvesting coupon payment
	112,83	proceeds received
P=	110,88	cost of the investment
	7,71%	implied repo rate