1. The Time Value of Money

1.1 Compounding and Discounting

Capitalization (compounding, finding future values) is a process of moving a value forward in time. It yields the future value given the relevant compounding rate (return rate, interest rate, growth rate).

Actualization (discounting, finding present values) is the reverse process. When we compute present values, we move backward in time. Discounting yields the present value of a future value given the relevant discounting rate (decline rate, interest rate, reduction rate).

Compounding (or discounting) is also a process of converting flow variables (such as sales, expenditures, cash flows, all items shown in income statement or cash flow statement)) into a stock variable (value, a balance sheet item) in two steps:

1. Identifying the cash flows provided by the asset
2. Compounding or discounting these cash flows at the appropriate growth or discounting rate.

Table 1. Future Value and Present Value Factors

Factor	Formula	Method of Calculation
Future value of a single sum, $FVF_{i,n}$	$(1+i)^n$	$(1+i)^n$
Present value of a single sum, $PVF_{i,n}$	$\frac{1}{(1+i)^n}$	$\frac{1}{(1+i)^n}$
Future value of an ordinary annuity, $FVFA_{i,n}$	$\sum_{t=1}^{n}(1+i)^{n-1}$	$\frac{(1+i)^n-1}{i}$
Future value of an annuity due	$(1+i)\sum_{t=1}^{n}(1+i)^{n-1}$	$\frac{(1+i)^n-1}{i}(1+i)$
Present value of an annuity, $PVFA_{i,n}$	$\sum_{t=1}^{n}\frac{1}{(1+i)^n}$	$\frac{1-\left[\frac{1}{(1+i)^n}\right]}{i}$
Present value of an annuity due	$(1+i)\sum_{t=1}^{n}\frac{1}{(1+i)^n}$	$(1+i)\frac{1-\left[\frac{1}{(1+i)^n}\right]}{i}$

1.2 A single cash flow

The future value factor (growth factor) is the future value of $1 at interest rate i for n periods. The future value of a single sum in n years is determined by

(1) $$FVF_{i,n} = (1+i)^n$$

The present value factor (discounting factor) is the present value of $1 received or paid at the end of the *n*th year with the rate i. Discounting is simply the reverse of compounding. The present value factor is equal to

(2) $$PVF_{i,n} = \frac{1}{(1+i)^n}$$

The future value (FV) is equal to the present value (PV) multiplied by the future value factor

(3) $$FV = PV(1+i)^n$$

The present value (PV) is equal to the future value multiplied by the present value factor

(4) $$PV = FV\frac{1}{(1+i)^n}$$

The rate (compounding or growth rate, discounting rate) is found as

(5) $$i = \sqrt[n]{\frac{FV}{PV}} - 1$$

Number of periods is given by

(6) $$n = \frac{\ln\frac{FV}{PV}}{\ln(1+i)}$$

Discrete Compounding

To reflect the frequency of compounding periods, two adjustments are required. First, the interest rate is converted to a per-period rate by dividing the annual rate by the number of compounding periods in a year. Second, the number of years is multiplied by the number of periods that occur each year, mn. The calculation of future value using discrete compounding is

(1) $$FV = PV\left(1+\frac{i}{m}\right)^{mn}$$

Continuous Compounding

If the compounding period approaches to zero, the future value is

(2) $$FV = \lim_{m\to\infty} PV\left(1+\frac{i}{m}\right)^{mn} = PVe^{in}$$

where e is Euler's constant, which is approximately 2,72.

1.3 Equal cash flows

An annuity is a series of level uninterrupted cash flows occurring at regular intervals. **Ordinary annuity (or deferred annuity)** is a sequence of uninterrupted, equal cash flows with payments (receipts) occurring at the end of each period. **Annuity due** is a sequence of uninterrupted, equal cash flows with payments (receipts) occurring at the beginning of each period.

The future value factor for an annuity determines the future value (at the end of period t=n) of the sum of capitalized cash flows of $1 received or paid at the end of each period (beginning from t=1) for a specified number of periods. It represents the sum of a series of future value single-sum factors:

$$\text{(7)} \qquad FVFA_{i,n} = (1+i)^{n-1} + (1+i)^{n-2} + ... + (1+i)^{1} + 1 = \frac{(1+i)^n - 1}{i}$$

The future value of annuity PMT (payment) at the end of n periods is found as

$$\text{(8)} \qquad FV = PMT\frac{(1+i)^n - 1}{i}$$

The future value of annuity due with the payments occurring at the beginning of each period can be calculated as

$$\text{(9)} \qquad FV = PMT\frac{(1+i)^n - 1}{i}(1+i)$$

The present value factor for an annuity is the sum of a series of present value factors for a single sum. It determines the present value (at the end of period t=0) of the sum of a series of discounted cash flows of $1 received or paid at the end of each period (beginning from t=1) for a specified number of periods. Thus,

$$\text{(10)} \qquad PVFA_{i,n} = \frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + ... + \frac{1}{(1+i)^n} = \frac{(1+i)^n - 1}{(1+i)^n i}$$

The present value of an ordinary annuity PMT (payment) received or paid at the end of n periods is found as

$$\text{(11)} \qquad PV = PMT\frac{(1+i)^n - 1}{(1+i)^n i}$$

The present value of an annuity due with the payments occurring at the beginning of each period can be calculated as

$$\text{(12)} \qquad PV = PMT\frac{(1+i)^n - 1}{(1+i)^n i}(1+i)$$

There are four variables in this equation. If you know the values of any three, you can find the value of the forth. If you know the present value, the rate and the number of periods, **the payment (ordinary annuity)** is

$$\text{(13)} \qquad PMT = PV\frac{(1+i)^n i}{(1+i)^n - 1}$$

If you know the future value, the rate and the number of periods, the payment (ordinary annuity) is

$$\text{(14)} \quad PMT = FV\frac{i}{(1+i)^n - 1}$$

A perpetuity is a series of equal payments that continue forever (to infinity). The present value of a perpetuity received or paid at the end of each period can be found with a formula:

$$\text{(15)} \quad PV = \lim_{n\to\infty} PMT\frac{(1+i)^n - 1}{(1+i)^n i} = \lim_{n\to\infty} PMT\frac{\frac{(1+i)^n}{(1+i)^n} - \frac{1}{(1+i)^n}}{\frac{(1+i)^n}{(1+i)^n} i},$$

or

$$\text{(16)} \quad PV = PMT\frac{1}{i}$$

The present value of a perpetuity received or paid at the beginning of each period can be found as follows:

$$\text{(17)} \quad PV = PMT\frac{1}{i}(1+i)$$

1.4 Growing Cash Flows

The general equation used to find the future value of an n-period growing annuity at a constant rate g is shown below:

$$\text{(18)} \quad \begin{aligned} FV &= PMT_1(1+i)^{n-1} + PMT_1(1+g)(1+i)^{n-2} + PMT_1(1+g)^2(1+i)^{n-3} + \ldots \\ &\ldots + PMT_1(1+g)^{n-2}(1+i)^1 + PMT_1(1+g)^{n-1} = PMT_1\frac{(1+i)^n - (1+g)^n}{i-g} \end{aligned}$$

The present value of an n-period annuity growing at a constant rate g is given by

$$\text{(19)} \quad PV = \frac{PMT_1}{i-g}\left[1 - \left(\frac{1+g}{1+i}\right)^n\right]$$

A **growing perpetuity** is one in which the cash flows increase each period by constant rate, that cash flow in any period is (1+g) times larger than the previous cash flow. This process results in equation

$$\text{(20)} \quad PV = \lim_{n\to\infty}\frac{PMT_1}{i-g}\left[1 - \left(\frac{1+g}{1+i}\right)^n\right] = \frac{PMT_1}{i-g}$$

1.5 Uneven Cash Flow Streams

In practice most investment or financial decisions involve uneven or nonconstant cash flows.

PV

The present value of uneven cash flows is found as the sum of the present values of the individual cash flows:

(21) $$PV = \frac{CF_1}{(1+i)^1} + \frac{CF_2}{(1+i)^2} + \ldots + \frac{CF_n}{(1+i)^n}$$

NPV

In practice positive cash flows (cash inflows) and negative cash flows (cash outflows) are usually distinguished. The Net Present Value (NPV) of a stream of cash flows is the difference between the present value of the inflows and the present value of the outflows, that is

(3) $$NPV = PV_{inflows} - Pv_{outflows}$$

The equation for the NPV is usually presented as follows:

(22) $$NPV = \frac{NCF_1}{(1+i)^1} + \frac{NCF_2}{(1+i)^2} + \ldots + \frac{NCF_n}{(1+i)^n}$$

Cash flows may be positive or negative. The rate is the same for all periods (**traditional approach**).

It is also possible to use different discounting rates for each period (**non-arbitrage approach**). The NPV is then calculated as

(23) $$NPV = \frac{NCF_1}{(1+i_1)^1} + \frac{NCF_2}{(1+i_2)^2} + \ldots + \frac{NCF_n}{(1+i_n)^n}$$

IRR

Internal rate of return (IRR) is the discounting rate that makes NPV=0, or, equivalently, the rate that makes the present value of inflows equal to the present value of outflows.

Finding IRR means solving the following equation:

(24) $$\frac{NCF_1}{(1+IRR)^1} + \frac{NCF_2}{(1+IRR)^2} + \ldots + \frac{NCF_n}{(1+IRR)^n} = 0$$

This equation is a polynomial of the *n*th degree. If cash flows occur over more than two periods, the IRR cannot be solved directly, and therefore the trial-and error method or interpolation method becomes necessary.

UAS

The uniform annual series (UAS) is an annuity payment that is equivalent in present value terms to an irregular cash flow pattern that occurs over the same time period. The UAS is used in special situations in capital budgeting. The UAS is an annuity payment (PMT) found by solving the equation:

$$(25)\qquad NPV = \frac{PMT}{(1+i)^1} + \frac{PMT}{(1+i)^2} + \ldots + \frac{PMT}{(1+i)^n} = PMT\frac{1-\left[\frac{1}{(1+i)^n}\right]}{i} = PMT\frac{(1+i)^n - 1}{i(1+i)^n}$$

Thus, the uniform annual series (PMT) is calculated as

$$(26)\qquad PMT = \frac{NPV}{\frac{(1+i)^n - 1}{(1+i)^n i}} = \frac{NPV(1+i)^n i}{(1+i)^n - 1}$$

<u>Questions:</u>

1. What is the compounding process ? What is discounting ?
2. What is the difference between simple interest and compound interest ?
3. What is the difference between an ordinary annuity and annuity due ?
4. What is the difference between an annuity and a perpetuity ?
5. Interpret the NPV.
6. Interpret the IRR.
7. Explain the UAS.